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Figures are at the end of the paper.

Taiji Variations: Yin and Yang in Multiple Dimensions

Cameron Browne

SWiSHzone.com Pty Ltd

The Basement, 33 Ewell St, Balmain, Australia, 2041

Cameron.Browne@swishzone.com

Abstract

This paper briefly describes the historical background and geometric construction of the Taiji diagram, commonly known as the yin yang symbol. Some examples of two-and-three dimensional variations are given based on alternative construction methods, including a simplified baseball curve with shape parameterized by a single variable. This curve is used to describe the standard Taiji ball, and introduce a new style of ball more in keeping with the Taiji's original design principles.

Keywords: Taiji diagram; Yin and yang; Simplified baseball curve; Fractal; Art.

1. Introduction

The *Taiji diagram* (described here as the *Taiji* for short) is a schematic representation of the relationship between yin and yang, two opposing but complementary forces of ancient Chinese philosophy. The *yin* aspect is dark, passive, feminine, corresponds to the moon, and is typically colored black or blue. The *yang* aspect is light, active, masculine, corresponds to the sun, and is typically colored white or red.

Figure 1 (left) shows the modern incarnation of the Taiji that will be familiar to most western readers. This figure consists of two nestled comma-shaped forms called *fish*, each with a head, eye and tail, that combine to form a circle. The yang fish, being the active, lighter, upward-seeking force, is usually shown head upwards (or at least on top). Clockwise rotation appears to be the modern convention, though this is not always the case. Various theories exist for the presence of the eyes: an artifact from the design's original geometric derivation; a symbol of the notion that each force contains a seed of the other; or simply a design feature.

The Taiji diagram dates back to Bronze Age China. The top left sub-diagram of Figure 1 shows the Li Kan circle, one of the earliest incarnations of the Taiji, and the top right sub-figure shows the Yellow River Map from the I Ching. The modern style of Taiji, which came about in the 12th century [1], is often called the *monad* or the *yin-yang symbol*. The bottom left design shows an early example of the modern style, and the bottom right sub-figure shows the simplified eyeless design that now adorns the South Korean flag.

2. Geometric construction

The overall design is described entirely by circular arcs and is enclosed by a circle of radius r, as shown in Figure 2. Each fish has a *head* of radius $r/2$ whose center lies on a line bisecting the circle, and an *eye* with coincident center and radius $r/8$ (this eye radius is a somewhat arbitrary value and may vary according to the designer's preference). That part of each fish not included in its head forms its *tail*.

Banchoff and Giblin define a *piecewise circular* (PC) curve as a finite sequence of circular arcs or line segments, with the end point of one arc coinciding with the beginning point of the next [2]. A PC curve is said to be *smooth* if the directed tangent line at the end of one arc coincides with the directed tangent line at the beginning of the next.

In the case of three-arc PC curves, Figure 3 shows the three basic types in which one circle encloses the other two: the cardioid, the fish, and the arbelos. Note that the fish has three different node types where arc segments meet: a smooth convex node, an inflection node, and a rhampoid cusp at the tip of the tail.

Graf [3] notes that both fish need not be the same size to fill the enclosing circle perfectly. Figure 7 (left) shows some of the infinite number of asymmetrical fish pairs that fit the outer circle. In each case the centers of the fish heads lie on the line bisecting the circle.

Moreover, Graf also observes that the perimeter of each fish is equal to the perimeter of the enclosing circle, adding to the attractive symmetry of the shape. This can be readily demonstrated as follows, where r_1 and r_2 are the radii of the two fish heads ($r_1 + r_2 = r$):

$$Perimeter = \frac{2\pi r}{2} + \frac{2\pi r_1}{2} + \frac{2\pi r_2}{2} \quad (1)$$

$$= \pi r + \pi r_1 + \pi(r - r_1) \quad (2)$$

$$= 2\pi r \quad (3)$$

Figure 7 (right) shows how the eyes can be widened and recursively exploited to give a fractal Taiji design. Perhaps some philosophical insight can be read into this infinite complementarity of the yin and yang aspects at all scales.

3. Rolling disk method

Figure 5 shows an alternative method for constructing each fish, using a circle of increasing radius tangent to the head of its partner and swept around 180°. This method was outlined by Aaron Isaksen in a second year assignment at the University of California at Berkeley [4]. Isaksen points out that the radius of the rolling circle at a given angle θ, swept around a head of radius $r/2$, is given by:

$$r_\theta = -\frac{r(\cos\theta - 1)}{2(\cos\theta + 3)} \tag{4}$$

This method can be used to convert the two-dimensional Taiji into a three-dimensional model by simply sweeping a sphere rather than a circle. Figure 6 (left) shows two polysphere fish obtained by sampling rolling spheres at increasing intervals. Figure 6 (right) shows a stereo pair of a model formed by merging the volume formed by one swept sphere and subtracting the volume formed by the other from a solid hemisphere, based on a design by Isaksen. Note that this model forms a single textured object (except for the floating eye); the yin aspect is visually separated from the yang aspect using only shadow. The models in Figure 6 were rendered using the POV-Ray ray tracer. Note that this is a cross-eyed stereo pair; to see the effect, focus on the smaller sphere and cross your eyes rather than diverging them.

We turn now to another three-dimensional description of the Taiji obtained by mapping it to a sphere. However, we must first introduce the related concept of the baseball curve.

4. Simplified baseball curves

The *baseball curve* is a simple closed curve on the surface of a sphere that separates the sphere into two identical (congruent) regions [5]. It derives its name from the fact that the seam on a baseball is defined by such a curve.

The exact curve used to describe the two leather patches that make up a baseball was invented by Elias Drake in the 1840s [6], improved by trial and error then patented by C. H. Jackson in the 1860s, then more recently analyzed in detail by mathematician Richard Thompson in 1998 [7].

The design of an actual baseball is no trivial task and involves a number of subtleties. However, an approximate visual description of the path formed by the seam on a baseball can be readily obtained using four circular arcs [8], as shown in Figure 7. Given four equally spaced points (a, b, c and d) that lie on the sphere and form a square on a plane bisecting it, a circular arc on the surface of the sphere with two of those points as end points is defined (left). This arc is reflected through the other two points (middle) then both arcs reflected in the other hemisphere and rotated (right). The arc segment end points match in position, direction and curvature, hence the final result is a single smoothly continuous curve on the surface of the sphere. In Banchoff & Giblin's terminology [2], this *simplified baseball curve* is a smooth four-arc PC space curve.

Although constrained to pass through the four fixed points, the shape of the simplified baseball curve can be conveniently controlled by a single *separation* parameter s, as shown in Figure 8. If the four fixed points are considered to lie on the sphere's equator, then angle s determines the distance from the midpoint of each circular arc segment to its nearest pole; a change in s is reflected equally in all four arcs.

For a given separation angle s in the range 0..90º, the location on a unit sphere of the arc's fixed end points a and b (assumed to lie on the xz plane) and its midpoint e are given by:

$$a_{x,y,z} = \left(\frac{-\sqrt{2}}{2}, \quad 0, \quad \frac{\sqrt{2}}{2} \right) \tag{5}$$

$$b_{x,y,z} = \left(\frac{\sqrt{2}}{2}, \quad 0, \quad \frac{\sqrt{2}}{2} \right) \tag{6}$$

$$e_{x,y,z} = \left(0, \quad -\cos(s), \quad \sin(s) \right) \tag{7}$$

The arc's circle may be readily obtained by finding the plane that passes through a, b and e, then defining two lines upon this plane that perpendicularly bisect the line segments $\overline{ae}$ and $\overline{be}$. The point at which the bisectors intersect will be the circle's center.

Figure 9 shows the shape of the curve at various separation angles; an actual baseball approximated with this method would have a separation angle of around s=20º. Shown below each example are what the surface patches would look like laid out flat. Note that the surface patches for s=0º (left) decrease to zero width at their waists, effectively creating two tailed shapes per patch.

5. Taiji balls

Martin Gardner noted in 1960 that the baseball curve could be used as the basis for a Taiji ball, in which the Taiji design covers the surface of a sphere [9]. Several such balls are now commercially available, for example the Yin Yang Ball from http://www.yinyangball.com/, all of which follow the shape of the simplified baseball curve with separation angle s=45º.

Figure 10 shows the standard Taiji ball from various angles. It bears an excellent resemblance to its two-dimensional counterpart when viewed from the optimal angle (left), however, becomes less recognizable when viewed from other angles (middle and right). The tails that can be seen from the optimal angle are just artifacts of perspective; the yin and yang shapes on this ball correspond to the s=45º surface patches shown in Figure 9 (bottom right) which allow two eyes per shape but no tail.

Figure 11 shows a suggested improvement to the Taiji ball that preserves the tails. This is achieved by basing the design on a simplified baseball curve of s=0º rather than s=45º, effectively yielding two tailed shapes per patch as noted in the previous section. The final design on the new ball is therefore composed of two yin shapes and two yang shapes, each with one eye and one tail, rather than the standard ball's single yin shape and single yang shape, each with two eyes and no tail. We believe that this new design is more in keeping with the Taiji's origins and intentions, and that the new Taiji ball more closely resembles its two-dimensional counterpart when viewed from suboptimal angles.

In addition, the eyes on the standard Taiji ball lie in a plane that bisects the ball, and become increasingly difficult to see as the viewer becomes more perpendicular to that plane. By contrast, the eyes on the new Taiji ball are more evenly spaced with their centers approximately forming a tetrahedron, hence it is more likely to get a good view of at least one eye from any angle.

Figure 12 shows a rack of Taiji pool balls in the new tailed design with fractal eyes. All Taiji ball figures were modeled and rendered using the POV-Ray ray tracer.

Closing on a whimsical note, Figure 13 (left) makes the obvious extension of the fish shape to a klein bottle (a topological construction that has no inside or outside [6]) by extending the fish's tail to penetrate its head and join smoothly along the seam of its eye. This raises the question of whether two klein fish (light and dark) may be interconnected to form a single klein figure. However, it turns out that joining each tail to the other's eye, as per Figure 13 (right), yields a false klein construction with distinct sides; the dark fish has a light inside and the light fish has a dark inside, hence the duality of the figure is maintained.

7. Conclusion

The Taiji diagram has undergone many changes over the centuries, but has retained its basic character. The elegance and simplicity of this design allow the derivation of many interesting variants, some of which are described above.

A simplified and parameterized baseball curve is related to the Taiji ball, in order to derive a new design that addresses some shortcomings with the standard design. It would be interesting to compare two physical models of these designs side-by-side.

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Figure Captions

Figure 1. The modern taiji (left) and some variations (right).

Figure 2. Geometric construction of the figure.

Figure 3. The three-piece PC curve types: cardioid, fish, and arbelos.

Figure 4. Asymmetrical fish fitting (left) and a fractal design (right).

Figure 5. Rolling a circle to construct a fish.

Figure 6. Extruded polyspheres (left) and a stereo pair of a carved figure (right).

Figure 7. Simplified baseball curve from reflections and rotations of a single circular arc.

Figure 8. Parameterization by separation angle s.

Figure 9. Simplified baseball stitching at s = 0º, 15º, 30º and 45º of separation, with the unfolded surface patches for each design shown.

Figure 10. The standard Taiji ball (s = 45º) viewed from the optimal angle (left) and two less interesting angles.

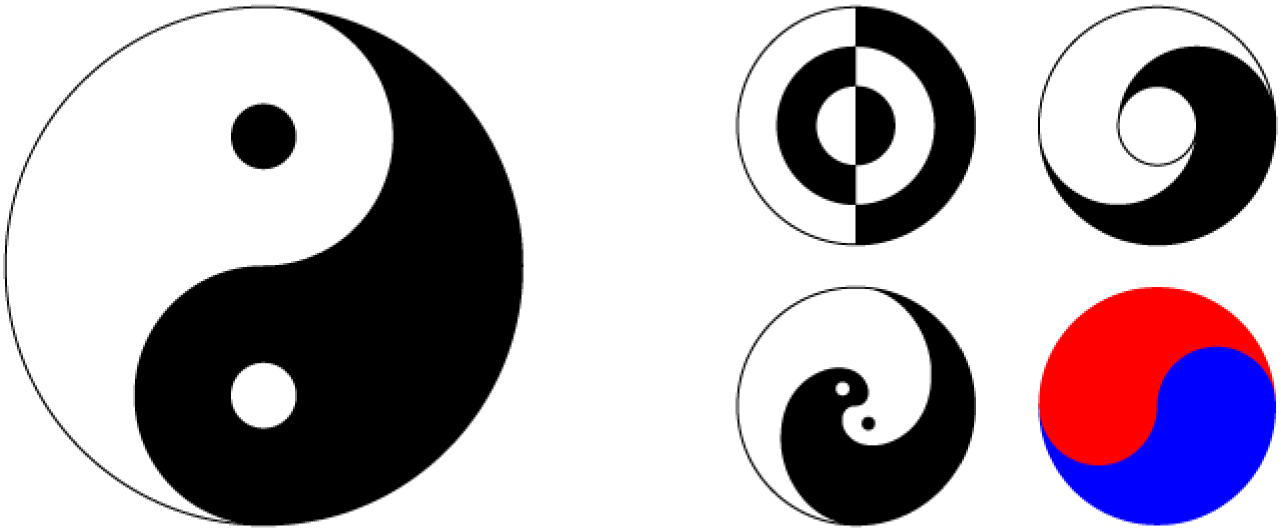
Figure 11. The new tailed Taiji ball (s = 0º).

Figure 12. A rack of tailed Taiji balls with fractal eyes.

Figure 13. A klein yang (left) and a false klein pair (right).

Figures for "Taiji variations"

Cameron Browne



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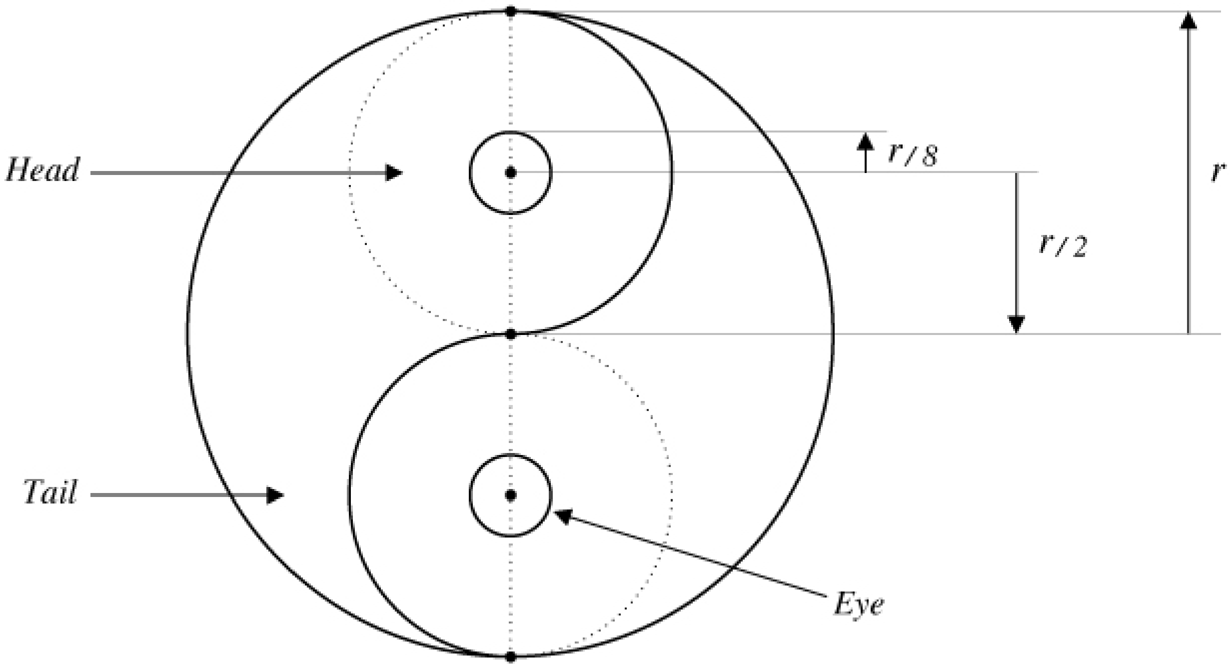

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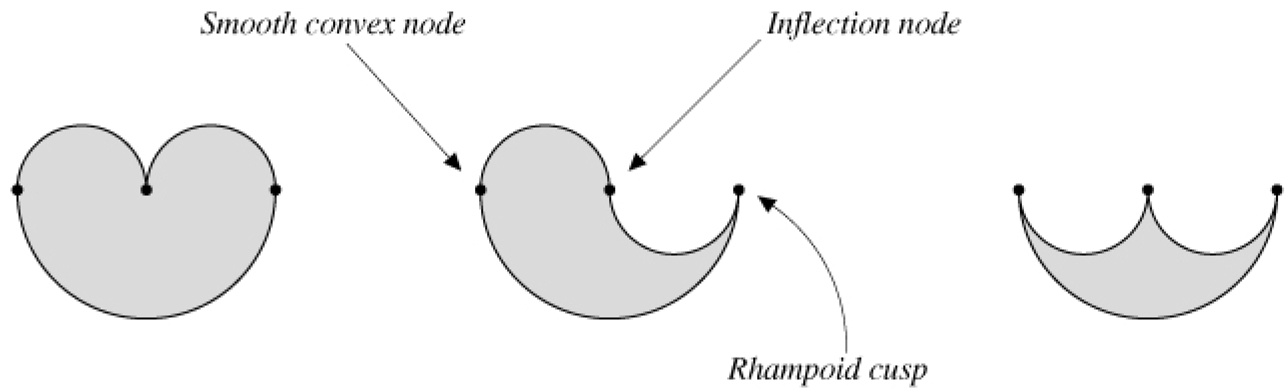

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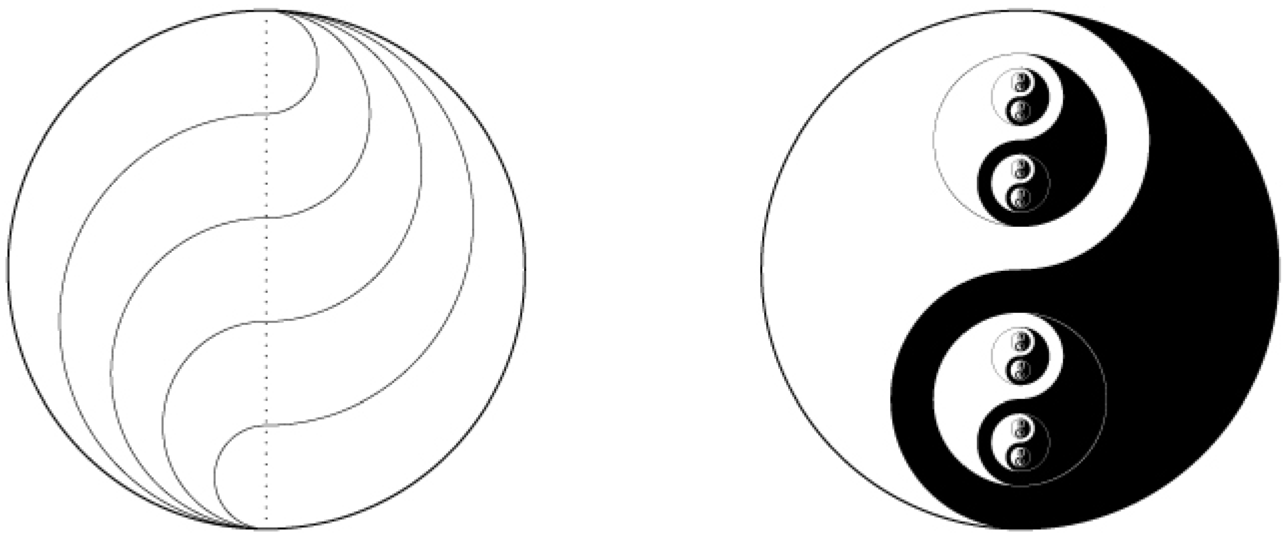
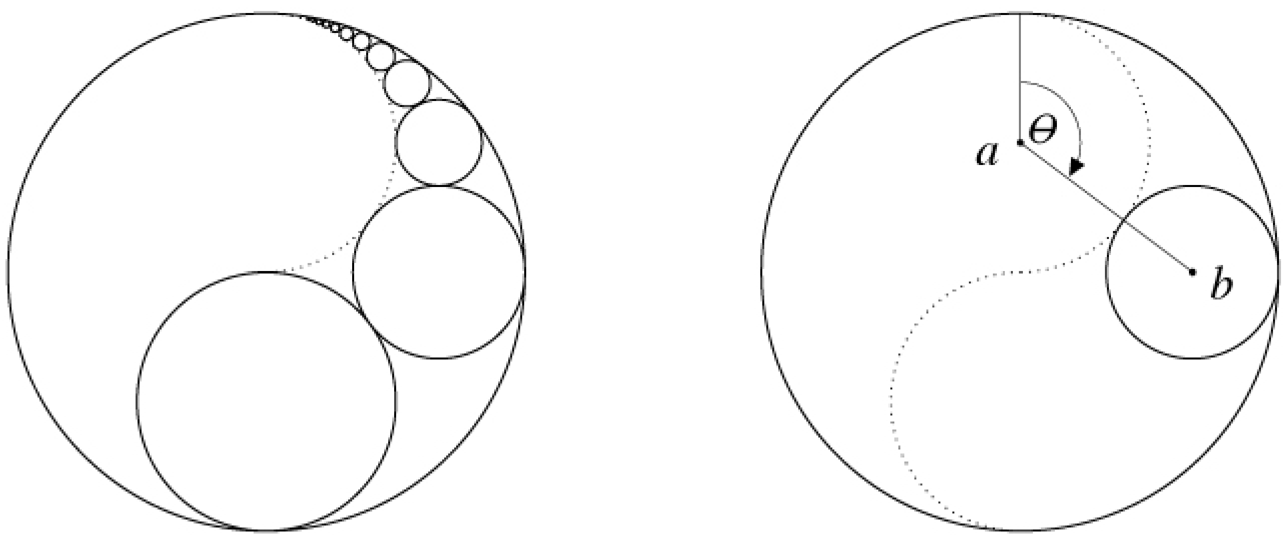


Figure 4. Asymmetrical fish fitting (left) and a fractal design (right).



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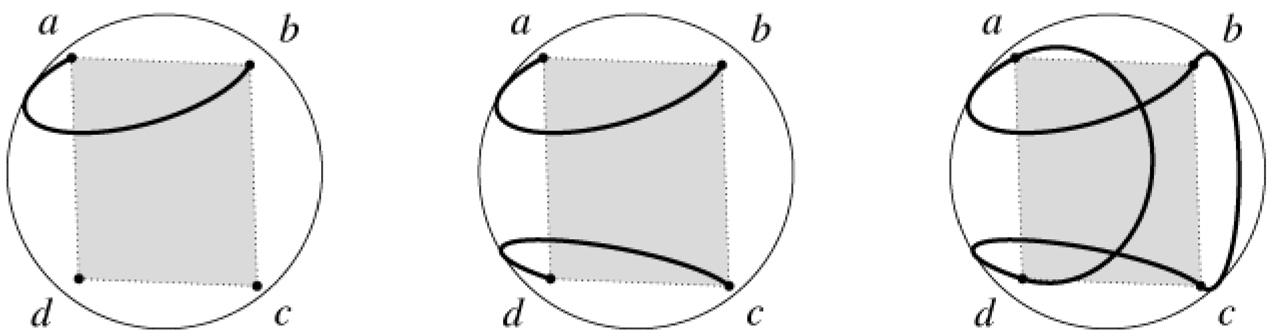

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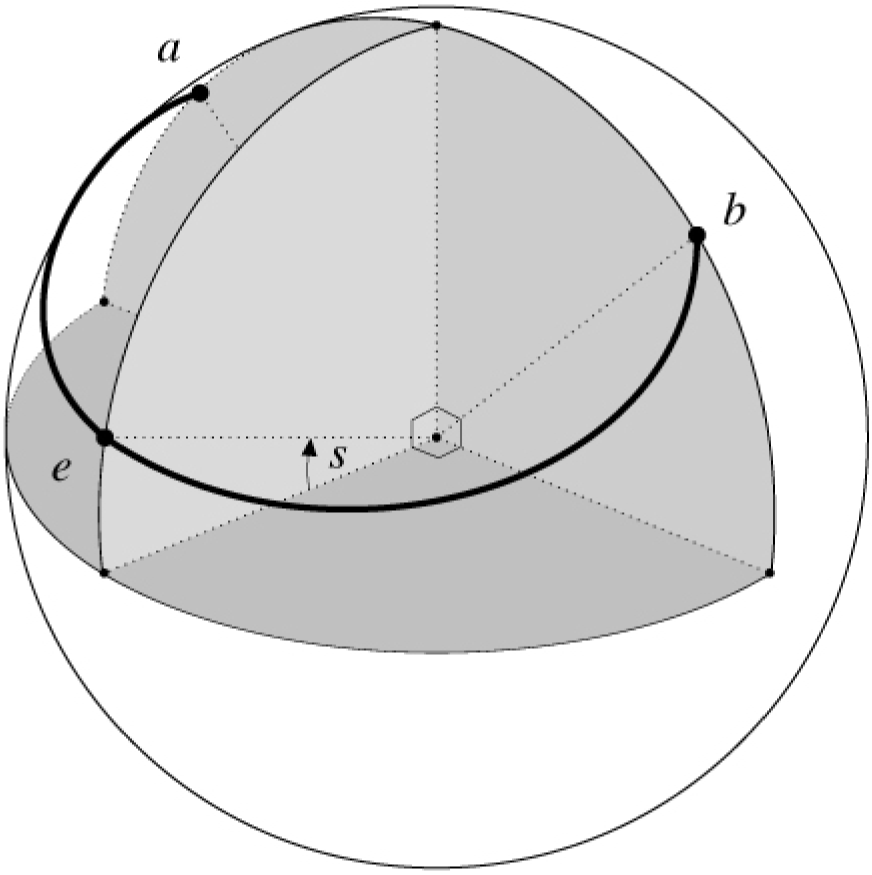
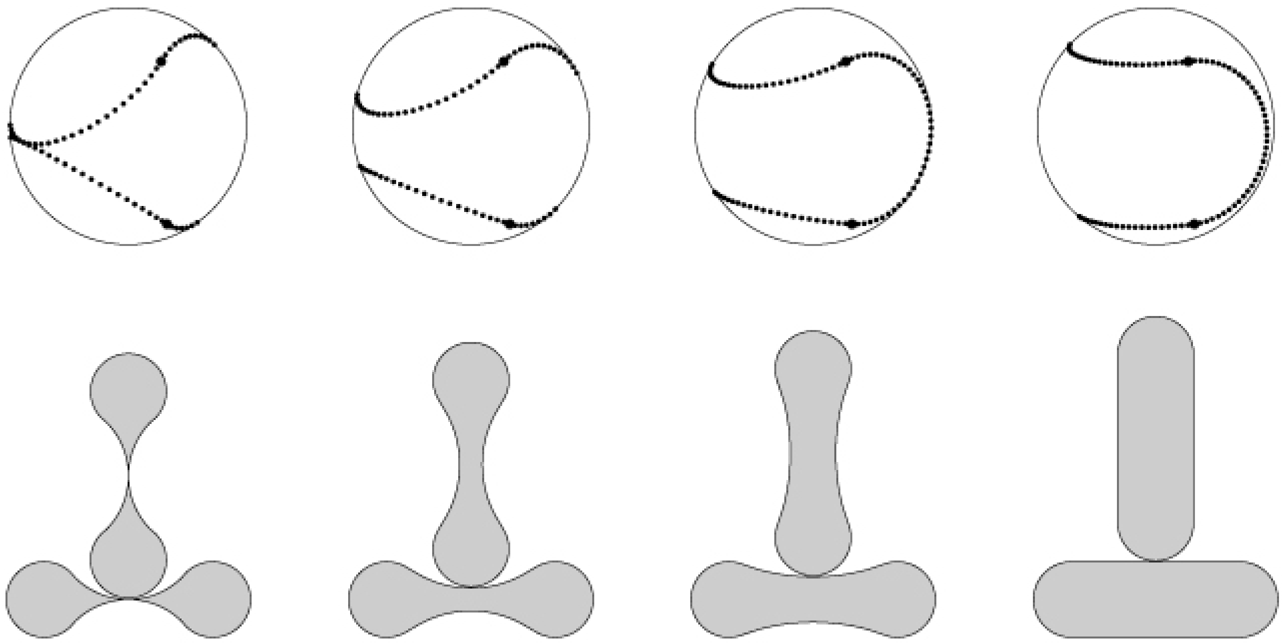


Figure 8. Parameterisation by separation angle ***s***.



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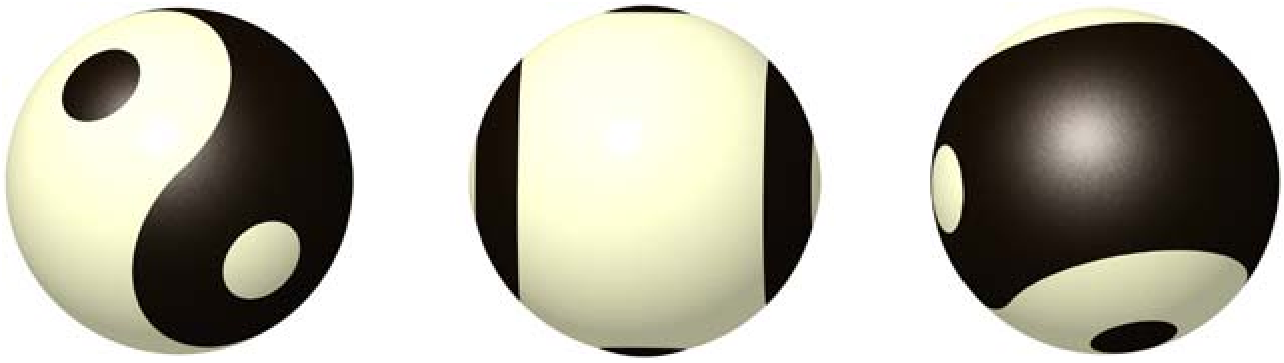

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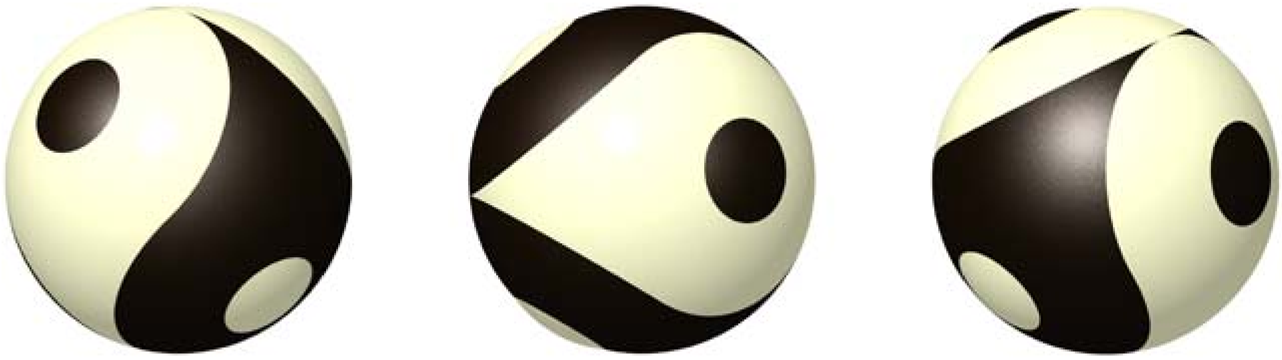

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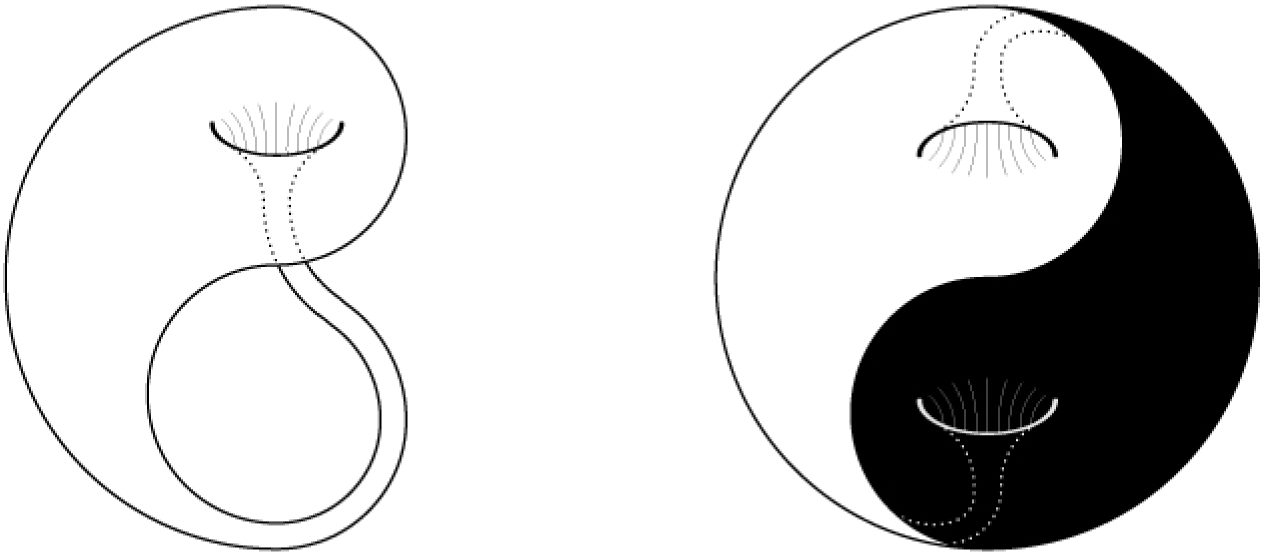

Figure 13. A klein yang (left) and a false klein pair (right).